Local restrictions to Asyomptotic Fermat with coefficients

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joint work with Luis Dieulefait

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Pitagorean triplet

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$(3, 4, 5)$ $(5, 12, 13)$ $(8, 15, 17)$ $(7, 24, 25)$ $(20, 21, 29)$ $(12, 35, 37)$ $(9, 40, 41)$ $(28, 45, 53)$ $(11, 60, 61)$ $(16, 63, 65)$ $(33, 56, 65)$ $(48, 55, 73), \ldots$

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The projective curve $C : X^2 + Y^2 = Z^2$ has genus 0. Since $C(\mathbb{Q}) \neq \emptyset$ then the set $C(\mathbb{Q})$ is infinite and admits a polynomial parametrization, over $\mathbb{Q}$.

Fermat equation

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Fermat (1637): "I can nicely prove that previous equation has no integer solution other than $(0, a, -a)$, $(a, 0, -a)$, $(a, -a, 0)$ but the margin in this Journal is so small..."

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Fermat (1637): "Cubum autem in duos cubos, aut quadratoquadratum in duos quadratoquadratos, et generaliter nullam in infinitum ultra quadratum potestatem in duos eiusdem nominis fas est dividere cuius rei demonstrationem mirabilem sane detexi, hanc marginis exiguitas non caperet."

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Theorem[Wiles, 1995]

Fermat last theorem is true.

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Proof.

Semistable rational elliptic curves are modular and so Fermat is true. □

Asymptotic Fermat with coefficients

Let a, b, c non-zero integers. We refer to the Diophantine equation

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as the Fermat equation of degree p with coefficients (a, b, c).

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What generalization should one expect?

Examples

The curve

$$(2^p + 1)X^p + Y^p + Z^p = 0$$

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Non - Theorem

There is a prime p_0 independent of (a, b, c) such that for every $p > p_0$ and every (a, b, c)

$$aX^p + bY^p + cZ^p = 0$$

has no projective points.

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$X^p + Y^p + 2Z^p = 0$ has solution $[1 : 1 : -1]$ for every p.

Asymptotic Fermat with coefficients

Conjecture[Kraus-Mazur]

Fix (a, b, c) integers. The set of solutions for previous equation for some prime $p \geq 5$ is finite in some sense.

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Fix (a, b, c) integers. The set

$$AF_{a,b,c} = \bigcup_{p \geq 5} C_{(a,b,c),p}(\mathbb{Q}) \subset \mathbb{P}_2(\mathbb{Q})$$

is finite.

Known cases

- Case $(a, b, c) = (1, 1, 1)$ was proved by A. Wiles.

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The problem becomes harder when trying to add more primes in a, b, c.

S-unit equations

Let S be a finite set of primes and let α, β, γ be non-zero integers. A solution to the S-unit equation

$$\alpha X + \beta Y + \gamma Z = 0$$

is a triple of nonzero integers (x, y, z) satisfying the equation such that $rad(xyz) = S$.

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Let $S = \emptyset$. Are there nonzero integers x, y, z such that

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Let $S = \{2, p\}$, $p \geq 3$. Has the S-unit equation $X + Y + Z = 0$ a solution?

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In this case it is possible. For instance $p = 2^n \pm 1$, i.e. p Mersenne or Fermat prime.

Theorem[Kraus,1997]

Let (a, b, c) be integers, coprime nonzero. Let S be the set of primes dividing abc. There is an S-unit equation $\alpha X + \beta Y + \gamma Z = 0$ depending on (a, b, c) such that if it has no solution then asymptotic Fermat Conjecture is true.

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Corollary 1

Asymptotic Fermat Conjecture $X^p + Y^p + Z^p = 0$ *is true for* p *big enough.*

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Corollary 2

Also if q is not Mersenne or Fermat prime then asymptotic Fermat conjecture $X^p + 2^s Y^p + q^r Z^p = 0$ is true.

Known facts on S-units

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S-unit equations are tough.

Case $|S| = 2$

Theorem 4 (Dieulefait, S.)

The equation

$$2^r q^s = \ell^{2t} - 1$$

for q, ℓ primes, r, s, t positive. Then the equation is one of the following

$$2^3 3 = 5^2 - 1$$
$$2^4 3 = 7^2 - 1$$
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The proof uses Catalan Conjeucture, Miahailescu's theorem.

The theorem

Theorem 5 (Dieulefait, S.)

Let n be an odd integer ≥ 3. Let $S = q_1 \cdots q_s$ be primes $q_i \equiv 1 (mod 4n)$. Assume that $rad(abc) = S$. Then the asymptotic Fermat conjecture

$$ax^p + by^p + cz^p = 0$$

is true.

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Let n be an odd integer ≥ 3. Let $S = q_1 \cdots q_s$ be primes $q_i \equiv 1 (mod 4n)$. Assume that $rad(abc) = S$. Then the asymptotic Fermat conjecture

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is true.

Proof.

The S-unit equation attached to (a, b, c) is $16X + Y + Z = 0$. It has no solution for a set S of primes $\equiv 1 \pmod{4n}$. □

A question

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has solution $9, 4, -25$.
Krauss theorem *does not apply*.

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Dieulefait & Soto: Solving $ax^p + by^p = cz^p$ with abc containing an arbitray number of prime factors, *arxiv*

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